- **Ideal or perfect fluid :-**

Frictionless, homogeneous and incompressible is one which is incapable of sustaining any tangential stress in the form of a shear but the normal force acts between the adjoining layers of the fluids. The pressure at every point of an ideal fluid is equal in all directions wheather the fluid be rest or in motion. This theory defines some concepts of the flow such as wave motion, the lift and the induced drag of an airfoil etc but it fails to define the fenomena such as skin friction, drag of a body etc.

- **Real or actual fluid :-**

Viscous and compressible is one in which both the tangential and normal forces exist. Viscosity, which is also known as an internal friction of a fluid is that characteristic of a real fluid which is capable of its offer resistance to shearing stresses. The resistance is small but not negligible for fluids such as water and gases but it is quite large for other fluids such as oil, glycerine, paints, col-tar etc. An incompressible fluid is one whose elements under go no change in volume or density. when a fixed mass of a fluid under goes in volume, its density also change. In other words, the ability for change in volume of a mass of fluid is known as compressibility.

- **Methods of describing a motion in a continuum :-**

There are two methods for studying the motion in a continuum mathematically. These are Lagrangian and Eulerian methods and referred to individual time rate rate of change and local time - rate of change respectively.

(i) Lagrangian Method :-

In this method, we study the history of each fluid particle i.e. any fluid particle is selected and each persued on its onward course observing the change in velocity, pressure and density at each point and at each its instant.

Let (x_0, y_0, z_0) be the co-ordinates of a chosen particle at a given time $t = t_0$. At a later time $t = t$ let the co-ordinate of the same particle be (x, y, z). since the chosen particle is any particle in continuum, the co-ordinates (x, y, z) will be the function of t and also of their initial values (x_0, y_0, z_0). Thus we have..

$$\left.\begin{aligned} x &= f_1(x_0, y_0, z_0, t) \\ y &= f_2(x_0, y_0, z_0, t) \\ z &= f_3(x_0, y_0, z_0, t) \end{aligned}\right\} \text{--- (i)}$$

Let u, v, w and a_x, a_y, a_z be the components of velocity and acceleration respectively. then we have --

$$u = \frac{\partial x}{\partial t}, \quad v = \frac{\partial y}{\partial t}, \quad w = \frac{\partial z}{\partial t} \quad \text{--- (ii)}$$

$$a_x = \frac{\partial^2 x}{\partial t^2}, \quad a_y = \frac{\partial^2 y}{\partial t^2}, \quad a_z = \frac{\partial^2 z}{\partial t^2} \quad \text{--- (iii)}$$

Note :- The fundamental equation of motion in Lagrangian form are non-linear and hence it leads to many difficulties while solving a problem. In fact, the present method is employed with an advantage only in some one-dimensional problems.

(ii) Eulerian Method :-

In this method, we select any point fixed in space ~~occup~~ occupied by the continuum and study the change which take place in velocity, pressure and density. as the continuum passes through this point. Let u, v, w be the components of velocity at the point (x, y, z) at any time t, then we have --

$$\left.\begin{aligned} u &= F_1(x, y, z, t) \\ v &= F_2(x, y, z, t) \\ w &= F_3(x, y, z, t) \end{aligned}\right\} \text{--- (iv)}$$

For a particular value of t. (iv) exhibits the motion at all points in the continuum at that time.

Note :- In Lagrangian method a particular fluid particle is identified and change in velocity etc are ~~to~~ studied as that fluid particle moves onwards. On the other hand in Eulerian method the individual particle is not identified. instant a point in the continuum is chosen and change in velocity etc. are studied as the continuum passes through the chosen fixed point.

• **Relation between the Lagrangian and Eulerian method:-**

• **Lagrangian to Eulerian :-**

Suppose $\phi(x_0, y_0, z_0, t)$ be some physical quantity involving Lagrangian description $\phi = \phi(x_0, y_0, z_0, t)$ ---- (i).

since the Lagrangian description i.e.

$$\left.\begin{aligned} x &= f_1(x_0, y_0, z_0, t) \\ y &= f_2(x_0, y_0, z_0, t) \\ z &= f_3(x_0, y_0, z_0, t) \end{aligned}\right\} \text{---- (ii)}$$

holds, solving (ii) for x_0, y_0, z_0 we get..

$$\left.\begin{aligned} x_0 &= g_1(x, y, z, t) \\ y_0 &= g_2(x, y, z, t) \\ z_0 &= g_3(x, y, z, t) \end{aligned}\right\} \text{---- (iii)}$$

using (iii), (i) becomes ----

$$\phi = \phi\{g_1(x, y, z, t), g_2(x, y, z, t), g_3(x, y, z, t), t\}$$

which expresses ϕ in terms of Eulerian ~~des~~ description.

• **Eulerian to Lagrangian :-**

Suppose $\psi(x, y, z, t)$ be some physical quantity involving Eulerian description

$\psi = \psi(x, y, z, t)$ .... (i)

Since Eulerian description

$$\left.\begin{aligned} u &= F_1(x, y, z, t) \\ v &= F_2(x, y, z, t) \\ w &= F_3(x, y, z, t) \end{aligned}\right\} \text{---- (ii)}$$

holds, $u = \frac{\partial x}{\partial t}$, $v = \frac{\partial y}{\partial t}$, $w = \frac{\partial z}{\partial t}$ -- (iii) holds for the proposed Lagrangian description.

Hence (i) and (ii) yields

$$\left.\begin{aligned} \frac{dx}{dt} &= F_1(x, y, z, t) \\ \frac{dy}{dt} &= F_2(x, y, z, t) \\ \frac{dz}{dt} &= F_3(x, y, z, t) \end{aligned}\right\} \text{--- (iv)}$$

The integration of (iv) involve three constants of integration which maybe taken as the initial co-ordinates (x_0, y_0, z_0) of the particle in a continuum. Thus a integration of (iv) leads to the well known equations of Lagrange i.e.

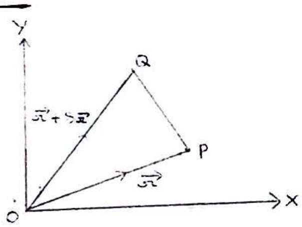
$$\left.\begin{aligned} x &= f_1(x_0, y_0, z_0, t) \\ y &= f_2(x_0, y_0, z_0, t) \\ z &= f_3(x_0, y_0, z_0, t) \end{aligned}\right\} \text{--- (v)}$$

using (v), (i) becomes

$$\psi = \psi\{f_1(x_0, y_0, z_0, t), f_2(x_0, y_0, z_0, t), f_3(x_0, y_0, z_0, t), t\}$$

which expresses ψ in terms of lagrangian description.

- **Velocity of a fluid particle :-**



Let the fluid particle be at P at any time t and let it be at Q at time $t + \delta t$. $\vec{OP} = \vec{r}$ and $\vec{OQ} = \vec{r} + \delta\vec{r}$, then in the interval δt the movement of the particle is $\vec{PQ} = \delta\vec{r}$.

Hence the particle velocity $\vec{q}$ at P is given by

$$\vec{q} = \lim_{\delta t \to 0} \frac{\delta\vec{r}}{\delta t} = \frac{d\vec{r}}{dt}$$

assuming that the above limit exists uniquely.

If u, v, w are the components of $\vec{q}$ along the axes then

$\vec{q} = u\hat{i} + v\hat{j} + w\hat{k}$

• Material, local and convective derivatives :-

Suppose a fluid particle moves from $P(x, y, z)$ at time t to $Q(x+\delta x, y+\delta y, z+\delta z)$ at time $t+\delta t$.

Further suppose that, $f(x, y, z, t)$ be a scaler function associated with some property of the fluid such as pressure, density etc. Let the total change of f due to the movement of the particle from P to Q be δf, then we have

$$\delta f = \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial z}\delta z + \frac{\partial f}{\partial t}\delta t$$

$$\Rightarrow \frac{\delta f}{\delta t} = \frac{\partial f}{\partial x}\frac{\delta x}{\delta t} + \frac{\partial f}{\partial y}\frac{\delta y}{\delta t} + \frac{\partial f}{\partial z}\frac{\delta z}{\delta t} + \frac{\partial f}{\partial t} \quad \text{--- (i)}$$

Let $\lim\limits_{\delta t \to 0} \frac{\delta f}{\delta t} = \frac{Df}{Dt}$, $\lim\limits_{\delta t \to 0} \frac{\delta x}{\delta t} = \frac{dx}{dt}$, $\lim\limits_{\delta t \to 0} \frac{\delta y}{\delta t} = \frac{dy}{dt}$, $\lim\limits_{\delta t \to 0} \frac{\delta z}{\delta t} = \frac{dz}{dt}$ (ii)

where $\vec{q} = (u, v, w)$ is the velocity of the fluid particle at P

Taking $\delta t \to 0$ in (i) and using (ii) we get.

$$\frac{Df}{Dt} = u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} + w\frac{\partial f}{\partial z} + \frac{\partial f}{\partial t} \quad \text{--- (iii)}$$

We have $\vec{q} = u\hat{i} + v\hat{j} + w\hat{k}$ --- (iv)

$$\vec{\nabla} = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k} \quad \text{(v)}$$

$$\therefore \vec{q}.\vec{\nabla} = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} \quad \text{--- (vi)}$$

using (vi), (iii) becomes ---

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + (\vec{q}\,\vec{\nabla})f$$

$$\Rightarrow \frac{D}{Dt} = \frac{\partial}{\partial t} + (\vec{q}.\vec{\nabla}) \quad \text{(vii)}$$

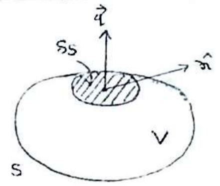
The operator $\frac{D}{Dt}$ is called the material derivative. The first term on the RHS of (vii) i.e. $\frac{\partial}{\partial t}$ is called the local derivative and it is associated with the time ~~var~~ variation at a fixed position. The 2nd term on the RHS of (vii) i.e. $\vec{q}.\vec{\nabla} = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$ is called the convective derivative and it it ~~is~~ is associated with the change of a physical quantity f due to the motion of the fluid particle.

- **The equation of continuity or the conservation of mass :-**

The law of conservation of mass states that the fluid mass can be neither created or destroyed. The equation of continuity aims at expressing the law of conservation of mass in a mathematical form.

Thus in a continuous motion the equation of continuity expresses the fact that the increase in the mass of the fluid ~~withou~~ within any closed surface drawn in the fluid in any time must equal to the excess of the mass that flows in over the mass that flows out

- **The equation of continuity by Euler's Method (Vector form) :-**



Let S be an arbitrary small closed surface drawn in the comprssible fluid enclosing a volume V and let S be the taken fixed in space. Let P(x, y, z) be any point on S and let $\rho(x, y, z, t)$ be the fluid density at P at any time t. Let δs denote the element of the surface S enclosing the point P. Let $\hat{n}$ be the unit ~~measure outward~~ outward do drawn normal at δs and $\vec{q}$ ~~from~~ is the fluid velocity at point P. Then the normal component of $\vec{q}$ measure outward from the volume V is $\hat{n}.\vec{q}$

Thus the rate of mass flow across $\delta s = \rho(\hat{n}.\vec{q}).\delta s$

The total ~~mass~~ rate of mass flow ~~ac~~ across $S = \int_S \rho(\hat{n}\,\vec{q})\,\delta s$

$$= \int_V \vec{\nabla}(\rho\vec{q})\,dV$$

The total rate of mass flow into $V = -\int_V \vec{\nabla}(\rho\vec{q})dV$ ---- (i)

Again the mass of the fluid within S at time $t = \int_V \rho\,dV$

The total rate of mass increases within $S = \frac{\partial}{\partial t}\int_V \rho\,dV$

$$= \int_V \frac{\partial \rho}{\partial t}\,dV \text{ ---- (ii)}$$

We suppose that the region V of the fluid contains no sources or sinks. Then by the law of conservation of the fluid mass, the rate of increase of the mass of the fluid within V must be equal to the total rate of mass flowing into V. Hence from (i) and (ii) we get.

$$\int_V \frac{\partial \rho}{\partial t}\, dV = -\int_V \vec{\nabla}(\rho\vec{q})\, dV$$

$$\Rightarrow \int_V \left\{\frac{\partial \rho}{\partial t} + \vec{\nabla}(\rho\vec{q})\right\} dV = 0$$

which holds for any arbitary small volume V. if

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}(\rho\vec{q}) = 0 \quad \text{--- (iii)}$$

Equation (iii) is called the equation of ~~continuity~~ continuity in Euler's vector form.

Note :- We have $\vec{\nabla}(\rho\vec{q}) = \rho\vec{\nabla}.\vec{q} + \vec{\nabla}.\rho\vec{q}$ ---- (iv)

using (iv), (iii) becomes....

$$\frac{\partial \rho}{\partial t} + \rho\vec{\nabla}.\vec{q} + \vec{\nabla}.\rho\vec{q} = 0$$

$$\Rightarrow \frac{D\rho}{Dt} + \rho\vec{\nabla}.\vec{q} = 0$$

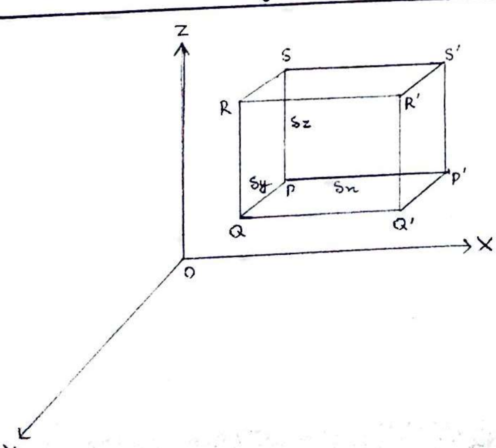
For an incompressible and heterogeneous fluid the density of any fluid particle is invariable with time and we get...

$\frac{D\rho}{Dt} = 0$ and the equation of continuity becomes. $\vec{\nabla}\vec{q} = 0$

$$\Rightarrow \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Again for an incompressible and homogeneous fluid ρ is constant and hence $\frac{\partial \rho}{\partial t} = 0$. then (iii) gives $\cancel{\vec{\nabla}(\rho q}$ $\vec{\nabla}.(\rho\vec{q}) = 0 \Rightarrow \vec{\nabla}\vec{q} = 0$

- **The equation of continuity in cartesian co-ordinate :-**



Let there be a fluid particle at $P(x, y, z)$, let $\rho(x, y, z, t)$ be the density of the fluid at P at any time t also let (u, v, w) be the velocity components at P parallel to the rectangular co-ordinate axes OX, OY, OZ respectively. We construct a small parallelopiped with edges of length $\delta x, \delta y, \delta z$ parallel to their respective co-ordinate axes having P at one of the angular point as shown in the figure then we have the mass of the fluid that passes through the face PQRS $= \rho(\delta y \delta z) u$ per unit time

$= f(x, y, z)$ (say) ... (i)

mass of the fluid that passes out through the face P'Q'R'S'

$= f(x+\delta x, y, z)$ per unit time

$= f(x, y, z) + \frac{\delta x}{1!} \frac{\partial}{\partial x} f(x, y, z) + \cdots$ (ii) [using Taylor's theorem]

∴ The net gain per unit time within the elegant due to flow through the faces PQRS and P'Q'R'S' = mass that enters through the face PQRS - mass that leaves through P'Q'R'S'

$= f(x, y, z) - \left[f(x, y, z) + \frac{\delta x}{1!} \frac{\partial}{\partial x} f(x, y, z) + \cdots\right]$

$= -\delta x \frac{\partial}{\partial x} f(x, y, z)$ [taking first order approximation]

$= -\delta x \frac{\partial}{\partial x}(\rho \delta y \delta z u)$

$= -\delta x \delta y \delta z \frac{\partial}{\partial x}(\rho u)$ ...... (iii)

Again, the net gain in mass per unit time within the element ~~elegant~~ due to flow through the faces PP'S'S and QQ'R'R is given by ---

$-\delta x \delta y \delta z \frac{\partial}{\partial y}(\rho v)$ ..... (iv)

and the net gain in mass per unit time within the element due to flow through PP'Q'Q and SS'R'R is given by --- $= -\delta x \delta y \delta z \frac{\partial}{\partial z}(\rho w)$ ... (v)

using (iii) (iv) and (v) we get the total rate of mass flow into the elementary parallelopiped $= -\delta x \delta y \delta z \left[\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w)\right]$ .... (vi)

Again the mass of the fluid within the chosen element at time t

$= \rho \delta x \delta y \delta z$

∴ The total rate of mass increase within the element

$$= \frac{\partial}{\partial t}(\rho\,\delta x\,\delta y\,\delta z) = \delta x\,\delta y\,\delta z\,\frac{\partial \rho}{\partial t} \quad \text{--- (vii)}$$

We suppose that the elementary parallelopiped of the fluid contains no source or sinks. then by the law of conservation of fluid mass the rate of increase of the mass of the fluid within the element must be equal to the rate of mass flowing into the element. Thus equation (vi) and (vii) are equal

$$\therefore\ \delta x\,\delta y\,\delta z\,\frac{\partial \rho}{\partial t} = -\,\delta x\,\delta y\,\delta z\left[\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w)\right]$$

$$\Rightarrow \left[\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w)\right]\delta x\,\delta y\,\delta z = 0$$

which holds for any arbitary small volume of the rectangular parallelopiped if

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0$$

$$\Rightarrow \frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0$$

$$\Rightarrow \frac{D\rho}{Dt} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0$$

which is the desire equation of continuity in cartesian co ordinate

• The velocity potential :-

Let the fluid velocity at time t is $\vec{q} = (u, v, w)$ also let at the considered instant t $\exists$ a scaler function $\phi(x, y, z, t)$ uniform throughout the entire fluid of flow s.t. $-d\phi = u\,dx + v\,dy + w\,dz$ --- (i)

$$\Rightarrow -\left(\frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz\right) = u\,dx + v\,dy + w\,dz \quad \text{(ii)}$$

Then the expression on the RHS of (i) is an exact differential and we have

$$u = -\frac{\partial \phi}{\partial x},\quad v = -\frac{\partial \phi}{\partial y},\quad w = -\frac{\partial \phi}{\partial z} \quad \text{--- (iii)}$$

$$\therefore \vec{q} = u\hat{i} + v\hat{j} + w\hat{k}$$

$$= -\frac{\partial \phi}{\partial x}\hat{i} - \frac{\partial \phi}{\partial y}\hat{j} - \frac{\partial \phi}{\partial z}\hat{k} = -\,\text{grad}\,\phi \quad \text{--- (iv)}$$

ϕ is called the velocity potential. the $-$ve sign in (iv) is a convention. It ensures that the flow takes place from the higher to lower potentials.

The necessary and sufficient ~~cont~~ condition for (iv) holds is

$\vec{\nabla} \times \vec{q} = 0 \Rightarrow \text{curl}\,\vec{q} = 0 \qquad \text{(v)}$

$$\Rightarrow \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix} = 0$$

$$\Rightarrow \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\hat{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\hat{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\hat{k} = 0$$

When equation (v) holds, the flow is known as irrotational.

For such flow the ~~fluid~~ field of the velocity $\vec{q}$ is conservative.

We know that the equation of continuity of an incompressible fluid

is $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$

Let the fluid ~~f~~ moves irrotational, then the velocity potential ϕ exists

and we have $u = -\frac{\partial \phi}{\partial x}$, $v = -\frac{\partial \phi}{\partial y}$, $w = -\frac{\partial \phi}{\partial z}$

then the equation of continuity becomes

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

$$\Rightarrow \vec{\nabla}^2 \phi = 0$$

$\Rightarrow \phi$ is a harmonic function because it satisfies the Laplace equation $\nabla^2\phi = 0$.

• Rotational and Irrotational motion :-

The motion of a fluid is said to be irrotational when the vorticity vector $\vec{\Omega}$ of every fluid particle is zero and when the vorticity vector is non-zero, the motion is said to be rotational.

We can conclude that the motion is irrotational if curl $\vec{q} = 0$

i.e. $\frac{\partial w}{\partial y} = \frac{\partial v}{\partial z}$, $\frac{\partial u}{\partial z} = \frac{\partial w}{\partial x}$, $\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}$

• The vorticity vector :-

Let $\vec{q} = u\hat{i} + v\hat{j} + w\hat{k}$ be the fluid velocity s.t. curl $\vec{q} \neq 0$, then the vector $\vec{\Omega}$ = curl $\vec{q}$ is called the vorticity vector.

Let $\Omega_x, \Omega_y, \Omega_z$ be the components of $\vec{\Omega}$

$$\text{curl}\,\vec{q} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\hat{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\hat{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\hat{k}$$

$$\Omega_x = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \quad \Omega_y = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \quad \Omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

• Test whether the motion specified by -- $\vec{q} = \dfrac{k^2(x\hat{j} - y\hat{i})}{x^2+y^2}$, k is constant is a possible motion for an incompressible fluid. Also test whether the motion is of the potential kind and if so, determine the velocity potential.

$\Rightarrow$ $\vec{q} = \dfrac{k^2(x\hat{j} - y\hat{i})}{x^2+y^2} = u\hat{i} + v\hat{j}$

$\therefore$ $u = -\dfrac{k^2 y}{x^2+y^2}$, $v = \dfrac{k^2 x}{x^2+y^2}$

$\therefore$ $\dfrac{\partial u}{\partial x} = (-k^2 y).\left(-\dfrac{1}{x^2+y^2} . 2x\right) = \dfrac{-2k^2xy}{x^2+y^2}$

$\dfrac{\partial v}{\partial y} = k^2 x\left(-\dfrac{1}{x^2+y^2} . 2y\right) = \dfrac{2k^2xy}{x^2+y^2}$

$\therefore$ $\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0$

$\therefore$ The motion is a possible motion for an incompressible fluid.

Now, $\text{curl}\,\vec{q} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -\dfrac{k^2 y}{x^2+y^2} & \dfrac{k^2 x}{x^2+y^2} & 0 \end{vmatrix}$

$= \hat{i}.0 + \hat{j}.0 + \hat{k}\left\{\dfrac{\partial}{\partial x}\left(\dfrac{k^2x}{x^2+y^2}\right) + \dfrac{\partial}{\partial y}\left(\dfrac{k^2y}{x^2+y^2}\right)\right\}$

$= \hat{k}.\,k^2\left\{\dfrac{1}{x^2+y^2} - \dfrac{2x^2}{(x^2+y^2)^2} + \dfrac{1}{x^2+y^2} - \dfrac{2y^2}{(x^2+y^2)^2}\right\}$

$= \hat{k}.\,k^2\left\{\dfrac{2}{x^2+y^2} - \dfrac{2}{x^2+y^2}\right\}$

$= 0$

$\therefore$ The flow is of potential kind.

Now we find the velocity potential $\phi(x,y,z)$ s.t.

$$\vec{q} = -\vec{\nabla}\phi = -\frac{\partial\phi}{\partial x}\hat{i} - \frac{\partial\phi}{\partial y}\hat{j} - \frac{\partial\phi}{\partial z}\hat{k}$$

$\therefore$ $\dfrac{\partial\phi}{\partial x} = \dfrac{k^2 y}{x^2+y^2}$ --- (i)

$\dfrac{\partial\phi}{\partial y} = -\dfrac{k^2 x}{x^2+y^2}$ --- (ii) $\qquad \dfrac{\partial\phi}{\partial z} = 0$ --- (iii) equation (iii) shows that the velocity potential ϕ is a function of x and y only. i.e $\phi = \phi(x, y)$

integrating (i) we get

$$\phi = k^2 \tan^{-1} \frac{x}{y} + F(y) \quad \text{--- (iv)}$$

where $F(y)$ is a arbitary function of y

$$\frac{\partial \phi}{\partial y} = -\frac{k^2 x}{x^2 + y^2} + F'(y) \quad \text{--- (v)}$$

comparing (ii) and (v) we get

$$F'(y) = 0 \Rightarrow F(y) = \text{constant}$$

since the constant can be omitted, let us take $F(y) = 0$ and we have the velocity potential as

$$\phi(x, y) = k^2 \tan^{-1} \frac{x}{y}$$

- **Give examples of irrotational and rotational flows.**

$\Rightarrow$ Let us take the fluid velocity as $\vec{q} = kx\hat{i} + 0\hat{j} + 0\hat{k}$

$$\text{curl } \vec{q} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ kx & 0 & 0 \end{vmatrix}$$

$$= 0$$

$\therefore$ The flow is irrotational.

Again let us take another velocity of p a particle

$\vec{q} = ky\hat{i} + 0\hat{j} + 0\hat{k}$ and k is non-zero.

$$\therefore \text{curl } \vec{q} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ky & 0 & 0 \end{vmatrix}$$

$$= -k\hat{k}$$

$\therefore$ The flow is rotational.

• The velocity components for a two - dimensional fluid system can be given in the Eulerian system by ---

$$u = 2x + 2y + 3t, \quad v = x + y + \frac{1}{2}t$$

Find the displacement of a fluid particle in lagrangian system.

⇒ Given that
$$\left.\begin{aligned} u &= 2x + 2y + 3t \\ v &= x + y + \frac{1}{2}t \end{aligned}\right\} \cdots (i)$$

in terms of the displacement x and y the velocity components u and v can be represented as ---
$$\left.\begin{aligned} u &= \frac{dx}{dt} \\ v &= \frac{dy}{dt} \end{aligned}\right\} \cdots (ii)$$

From (i) and (ii) we get ---

$$\left.\begin{aligned} \frac{dx}{dt} &= 2x + 2y + 3t \\ \frac{dy}{dt} &= x + y + \frac{1}{2}t \end{aligned}\right\} \cdots (iii)$$

Let $D \equiv \frac{d}{dt}$ then equation (iii) becomes ..

$$(D-2)x - 2y = 3t \quad \cdots (iv)$$

$$-x + (D-1)y = \frac{1}{2}t \quad \cdots (v)$$

(v) × (D-2) and adding with (iv) we get.

$$-2y + (D-2)(D-1)y = 3t + (D-2)\frac{1}{2}t$$

$$\Rightarrow (D^2 - 3D)y = 3t + \frac{1}{2} - \frac{1}{2}t$$

$$\Rightarrow D(D-3)y = 2t + \frac{1}{2} \quad \cdots (vi)$$

A.E. is ... $m^2 - 3m = 0 \Rightarrow m = 0, 3$

$$CF = c_1 + c_2 e^{3t}$$

$$\therefore PI = \frac{1}{D(D-3)}\left(2t + \frac{1}{2}\right)$$

$$= \frac{1}{D}\left(-\frac{1}{3}\right)\left(1 - \frac{D}{3}\right)^{-1}\left(2t + \frac{1}{2}\right)$$

$$= \frac{1}{D}\left(-\frac{1}{3}\right)\left(1 + \frac{D}{3} + \cdots\right)\left(2t + \frac{1}{2}\right)$$

$$= \frac{1}{D}\left(-\frac{1}{3}\right)\left(2t + \frac{1}{2} + \frac{2}{3}\right)$$

$$= \frac{1}{D}\left(-\frac{2t}{3} - \frac{7}{18}\right) = -\frac{t^2}{3} - \frac{7}{18}t$$

$\therefore$ The G.S is $y = c_1 + c_2 e^{3t} - \frac{t^2}{3} - \frac{7t}{18}$ ..... (vii)

$$\frac{dy}{dt} = 3c_2 e^{3t} - \frac{2}{3}t - \frac{7}{18}$$

$$x = \frac{dy}{dt} - y - \frac{1}{2}t$$

$$= 3c_2 e^{3t} - \frac{2}{3}t - \frac{7}{18} - c_1 - c_2 e^{3t} + \frac{t^2}{3} + \frac{7t}{18} - \frac{1}{2}t$$

$$= -c_1 + 2c_2 e^{3t} + \frac{1}{3}t^2 - \frac{7}{9}t - \frac{7}{18} \quad \text{(viii)}$$

Now we can use the following initial conditions $x = x_0,\ y = y_0,\ t = t_0 = 0$ --- (ix)

using this (vii) and (viii) becomes ---

$$y_0 = c_1 + c_2$$

$$x_0 = -c_1 + 2c_2 - \frac{7}{18}$$

solving we get --- $c_1 = \frac{2y_0 - x_0}{3} - \frac{7}{54}$

$$c_2 = \frac{x_0 + y_0}{3} + \frac{7}{54}$$

putting this in (vii) and (viii) we get.

$$x = -\left(\frac{2y_0 - x_0}{3} - \frac{7}{54}\right) + 2\left(\frac{x_0 + y_0}{3} + \frac{7}{54}\right)e^{3t} + \frac{1}{3}t^2 - \frac{7}{9}t - \frac{7}{18} \quad \text{--- (ix)}$$

$$y = \left(\frac{2y_0 - x_0}{3} - \frac{7}{54}\right) + \left(\frac{x_0 + y_0}{3} + \frac{7}{54}\right)e^{3t} - \frac{1}{3}t^2 - \frac{7}{18}t \quad \text{--- (x)}$$

(ix) and (x) give the desire displacement x and y in Lagrangian system involving the initial positions $x = x_0,\ y = y_0$ and $t = 0$.

- Assuming that the velocity componants for a two - dimensional flow system can be given in the Eulerian system as ---

$$u = A(x+y) + ct$$

$$v = B(x+y) + Et$$

Find the displacement of a fluid particle in lagrangian system.

$\Rightarrow$ Given that
$$\left.\begin{aligned} u &= A(x+y)+ct \\ v &= B(x+y)+Et \end{aligned}\right\} \text{--- (i)}$$

in terms of the displacement x and y the velocity components.

u and v can be represented as ... $u = \frac{dx}{dt} = Dx$

$$v = \frac{dy}{dt} = Dy \quad \text{where } D \equiv \frac{d}{dt}$$
--- (ii)

From (i) and (ii) we get.

$$(D-A)x - Ay = ct \text{ --- (iii)}$$

$$-Bx + (D-B)y = Et \text{ --- (iv)}$$

(iii) $\times B$ + (iv) $\times (D-A)$ we get.

$$-ABy + (D-A)(D-B)y = Bct + (D-A)Et$$

$$\Rightarrow \{D^2 - (A+B)D\}y = Bct + E - AEt$$

$$\Rightarrow D\{D-(A+B)\}y = (Bc-AE)t + E \text{ --- (v)}$$

$\therefore$ AE. is ..... $m^2 - (A+B)m = 0 \Rightarrow m = 0, A+B$

$$\therefore \text{C.F.} = c_1 + c_2 e^{(A+B)t}$$

$$\therefore \text{P.I.} = \frac{1}{D(D-(A+B))}\{(Bc-AE)t+E\}$$

$$= \frac{1}{D}\left(-\frac{1}{A+B}\right)\left(1-\frac{D}{A+B}\right)^{-1}\{(Bc-AE)t+E\}$$

$$= \frac{1}{D}\left(-\frac{1}{A+B}\right)\left(1+\frac{D}{A+B}+\cdots\right)\{(Bc-AE)t+E\}$$

$$= \frac{1}{D}\left(-\frac{1}{A+B}\right)\left((Bc-AE)t+E+\frac{Bc-AE}{A+B}\right)$$

$$= \frac{1}{D}\left(-\frac{Bc-AE}{A+B}t - \frac{Bc+BE}{(A+B)^2}\right)$$

$$= -\frac{Bc-AE}{A+B}\cdot\frac{t^2}{2} - \frac{Bc+BE}{(A+B)^2}t$$

$\therefore$ The G.S is ... $y = c_1 + c_2 e^{(A+B)t} - \frac{Bc-AE}{A+B}\frac{t^2}{2} - \frac{Bc+BE}{(A+B)^2}t$ ---- (vi)

$$\frac{dy}{dt} = c_2 (A+B) e^{(A+B)t} - \frac{BC-AE}{A+B} t - \frac{BC+BE}{(A+B)^2}$$

$$Bx = \frac{dy}{dt} - By - Et$$

$$= c_2 (A+B) e^{(A+B)t} - \frac{BC-AE}{A+B} t - \frac{BC+BE}{(A+B)^2} - Bc_1 - Bc_2 e^{(A+B)t}$$

$$+ B \frac{BC-AE}{A+B} \frac{t^2}{2} + B \frac{BC+BE}{(A+B)^2} t - Et$$

$$= -Bc_1 + Ac_2 e^{(A+B)t} + B \frac{BC-AE}{A+B} \cdot \frac{t^2}{2} - \frac{ABC+ABE}{(A+B)^2} t - \frac{BC+BE}{(A+B)^2}$$

$$\Rightarrow x = -c_1 + \frac{A}{B} c_2 e^{(A+B)t} + \frac{BC-AE}{A+B} \cdot \frac{t^2}{2} - \frac{AC+AE}{(A+B)^2} t - \frac{C+E}{(A+B)^2} \quad \text{---} \; (vii)$$

Here we use the following initial conditions ~~as~~ $x = x_0$, $y = y_0$, $t = t_0 = 0$

Using this (v) and (vii) Becomes . .

$$y_0 = c_1 + c_2$$

$$x_0 = -c_1 + \frac{A}{B} c_2 - \frac{C+E}{(A+B)^2}$$

solving we get ----

$$c_1 = \frac{Ay_0 - Bx_0}{A+B} - \frac{B(C+E)}{(A+B)^3}$$

$$c_2 = \frac{B(x_0+y_0)}{A+B} + \frac{B(C+E)}{(A+B)^3}$$

putting this in (vi) and (vii) we get ....

$$x = -\left\{ \frac{Ay_0 - Bx_0}{A+B} - \frac{B(C+E)}{(A+B)^3} \right\} + A \left\{ \frac{x_0+y_0}{A+B} + \frac{C+E}{(A+B)^3} \right\} e^{(A+B)t}$$

$$+ \frac{BC-AE}{A+B} \frac{t^2}{2} - \frac{AC+AE}{(A+B)^2} t - \frac{C+E}{(A+B)^2} \quad \cdots \; (viii)$$

$$y = \left\{ \frac{Ay_0 - Bx_0}{A+B} - \frac{B(C+E)}{(A+B)^3} \right\} + \left\{ \frac{B(x_0+y_0)}{A+B} + \frac{B(C+E)}{(A+B)^3} \right\} e^{(A+B)t}$$

$$- \frac{BC-AE}{A+B} t^2/2 - \frac{BC+BE}{(A+B)^2} t \quad \cdots \; (ix)$$

(viii) and (ix) give the desire displacement x and y in Lagrangian system involving the initial positions $x = x_0$, $y = y_0$ and $t = 0$

• For a two - dimentional flow, the velocities at a point in a fluid may be expressed in the Eulerian co-ordinates by.... $u = x+y+2t$, $v = 2y+t$. Determine the Lagrange co-ordinates as functions of the initial positions x_0, y_0 and time $t=0$

⇒ Given that ....

$$\left.\begin{aligned} u &= x+y+2t \\ v &= 2y+t \end{aligned}\right\} \cdots (i)$$

in terms of the displacement x and y the velocity components.

u and v can be expressed as ...

$$\left.\begin{aligned} u &= \frac{dx}{dt} \\ v &= \frac{dy}{dt} \end{aligned}\right\} \cdots (ii)$$

From (i) and (ii) we get ....

$$\frac{dx}{dt} = x+y+2t \quad \cdots (iii)$$

$$\frac{dy}{dt} = 2y+t \quad \cdots (iv)$$

From (iv) we get ....

$$\frac{dy}{dt} - 2y = t$$

$$I.F = e^{\int -2dt} = e^{-2t}$$

multiplying by IF and integrating we get.

$$ye^{-2t} = c_1 + \int t e^{-2t} dt$$

$$\Rightarrow ye^{-2t} = c_1 - \frac{1}{4}(2t+1)e^{-2t}$$

$$\Rightarrow y = c_1 e^{2t} - \frac{1}{4}(2t+1) \quad \cdots (v)$$

putting the value of y in (iii) we get.

$$\frac{dx}{dt} - x = c_1 e^{2t} + \frac{1}{4}(6t-1)$$

$$\therefore I.F. = e^{\int -dt} = e^{-t}$$

multiplying by IF and integrating we get.

$$xe^{-t} = c_2 + \int e^{-t}\left[c_1 e^{2t} + \frac{1}{4}(6t-1)\right] dt$$

$$\Rightarrow xe^{-t} = c_2 + c_1 e^{t} - \frac{1}{4}(6t-1)e^{-t} - \frac{6}{4}e^{-t}$$

$$\Rightarrow x = c_2 e^{t} + c_1 e^{2t} - \frac{1}{4}(6t+5) \quad \cdots (vi)$$

Now we use the following initial conditions $x = x_0$, $y = y_0$ and $t = 0$,

using this (V) and (Vi) becomes

$$y_0 = c_1 - \frac{1}{4} \Rightarrow c_1 = y_0 + \frac{1}{4}$$

$$x_0 = c_1 + c_2 - \frac{5}{4}$$

$$\Rightarrow c_2 = x_0 - y_0 - \frac{1}{4} + \frac{5}{4} = x_0 - y_0 + 1$$

putting this in (V) and (V.) we get ---

$$x = (x_0 - y_0 + 1)e^{t} + \left(y_0 + \frac{1}{4}\right)e^{2t} - \frac{1}{4}(6t+5) \quad \text{---- (vii)}$$

$$y = \left(y_0 + \frac{1}{4}\right)e^{2t} - \frac{1}{4}(2t+1) \quad \text{--- (viii)}$$

(vii) and (viii) give the desire displacement x and y in lagrangian system

involving the initial positions $x = x_0$, $y = y_0$ and $t = 0$

- The velocities at a point in a fluid in the Eulerian system are given by--

$u = x+y+z+t$, $v = 2(x+y+z)+t$, $w = 3(x+y+z)+t$, obtain the displacement of a fluid particle in lagrangian system.

$\Rightarrow$ Given that

$$\left.\begin{aligned} u &= x+y+z+t \\ v &= 2(x+y+z)+t \\ w &= 3(x+y+z)+t \end{aligned}\right\} \text{--- (i)}$$

in terms of the displacement x, y and z, the velocity components.

u, v and w can be expressed as --

$$\left.\begin{aligned} u &= \frac{dx}{dt} = Dx \\ v &= \frac{dy}{dt} = Dy \\ w &= \frac{dz}{dt} = Dz \end{aligned}\right\} \text{--- (ii)} \quad \text{where } D \equiv \frac{d}{dt}$$

using (ii), (i) becomes

$$\frac{dx}{dt} = x+y+z+t \quad \text{---- (iii)}$$

$$\frac{dy}{dt} = 2(x+y+z)+t \quad \text{--- (iv)}$$

$$\frac{dz}{dt} = 3(x+y+z)+t \quad \text{--- (v)}$$

From (iii) and (iv)

$$(D-1)x - y = z+t \quad \text{--- (vi)}$$

$$-2x + (D-2)y = 2z+t \quad \text{--- (vii)}$$

(vi) × (D-2) and adding with (vii) we get...

$$(D-2)(D-1)x - 2x = Dz \cancel{+1} - 2z + 1 - 2t + 2z + t$$

$$\Rightarrow (D^2 - 3D)x = Dz + 1 - t \quad \text{.... (viii)}$$

Now, multiplying both sides of (vi) by 2 and (vii) by (D-1) and adding we get

$$-2y + 2(D-1)x - 2(D-1)x + (D-1)(D-2)y = 2z + 2t + 2Dz - 2z + 1 - t$$

$$\Rightarrow -2y + (D^2 - 3D + 2)y = 2Dz + 1 + t$$

$$\Rightarrow (D^2 - 3D)y = 2Dz + 1 + t \quad \text{.... (ix)}$$

From (v) we get.

$$(D-3)z = 3x + 3y + t$$

$$(D^2-3D)(D-3)z = 3(D^2-3D)x + 3(D^2-3D)y + (D^2-3D)t$$

$$\Rightarrow (D^3 - 6D^2 + 9D)z = 3Dz + 3 - 3t + 6Dz + 3 + 3t - 3$$

$$\Rightarrow (D^3 - 6D^2)z = 3 \quad \text{.... (x)}$$

A.E. is .. $m^3 - 6m^2 = 0 \Rightarrow m = 0, 0, 6$

∴ C.F is $c_1 + c_2 t + c_3 e^{6t}$

∴ P.I. $= \dfrac{1}{D^3 - 6D^2} 3$

$= -3 \dfrac{1}{6D^2(1 - D/6)} 1$

$= -\dfrac{1}{2D^2}\left(1 + \dfrac{D}{6} + \cdots\right) 1$

$= -\dfrac{1}{2D^2} \cdot 1 = -\dfrac{1}{2} \cdot \dfrac{t^2}{2}$

$= -\dfrac{1}{4} t^2$

∴ The G.S is -

$$z = c_1 + c_2 t + c_3 e^{6t} - \frac{1}{4} t^2$$

From (iv) and (v). we get ..

$$(D-2)y - 2z = 2x + t \quad \text{..... (xi)}$$

$$-3y + (D-3)z = 3x + t \quad \text{..... (xii)}$$

Now (xi) × (D-3) + (xii) × 2 we get ...

$$(D-3)(D-2)y - 2(D-3)z \cancel{6} - 6y + 2(D-3)z = 2Dx - 6x + 1 - 3t + 6x + 2t$$

$$\Rightarrow (D^2 - 5D)y = 2Dx + 1 - t \quad \text{--- (xiii)}$$

Again (xi) × 3 + (xii) × (D-2) we get.

$$3(D-2)y - 6z - 3(D-2)y + (D-2)(D-3)z = 6x + 3t + 3Dx + 1 - 6x - 2t$$

$$\Rightarrow (D^2 - 5D)z = 3Dx + 1 + t \quad \text{--- (xiv)}$$

From (ii) we get ...

$$(D-1)x = y + z + t$$

$$(D^2-5D)(D-1)x = (D^2-5D)y + (D^2-5D)z + (D^2-5D)t$$

$$\Rightarrow (D^3 - 6D^2 + 5D)x = 2Dx + 1 - t + 3Dx + 1 + t - 5$$

$$\Rightarrow (D^3 - 6D^2)x = -3$$

A.E. is ... $m^3 - 6m^2 = 0 \Rightarrow m = 0, 0, 6$

C.F. is ... $a_1 + a_2 t + a_3 e^{6t}$

$$\therefore \text{P.I.} = \frac{1}{D^3 - 6D^2}(-3) = (-3)\left(-\frac{1}{6}\right)\frac{1}{D^2\left(1 - \frac{D}{6}\right)} \cdot 1$$

$$= \frac{1}{2}\,\frac{1}{D^2}\left(1 - \frac{D}{6}\right)^{-1} 1$$

$$= \frac{1}{2}\,\frac{1}{D^2}\left(1 + \frac{D}{6} + \cdots\right) = \frac{1}{2}\,\frac{1}{D^2} \cdot 1$$

$$= \frac{1}{4}t^2$$

$\therefore$ The G.S. is ... $x = a_1 + a_2 t + a_3 e^{6t} + \frac{1}{4}t^2$

From (v) and (iii) we get ...

$$(D-3)z - 3x = 3y + t \quad \text{--- (xv)}$$

$$-z + (D-1)x = y + t \quad \text{--- (xvi)}$$

(xv) × (D-1) + ~~3×(x~~ (xvi) × 3 we get ...

$$(D-1)(D-3)z - 3(D-1)x - 3z + 3(D-1)x = 3Dy - 3y + 1 - t + 3y + 3t$$

$$\Rightarrow (D^2 - 4D)z = 3Dy + 1 + 2t \quad \text{--- (xvii)}$$

Again $(xv)\times 1 + (xvi)\times(D-3)$ we get...

$$(D-3)z - 3x - (D-3)z + (D-3)(D-1)x = 3y + t + Dy + 1 - 3y - 3t$$

$$\Rightarrow (D^2-4D)x = Dy + 1 - 2t \quad \ldots (xviii)$$

From (iv) we get

$$(D-2)y = 2x + 2z + t$$

$$\Rightarrow (D^2-4D)(D-2)y = 2(D^2-4D)x + 2(D^2-4D)z + (D^2-4D)t$$

$$\Rightarrow (D^3-6D^2+8D)y = 2Dy + 2 - 4t + 6Dy + 2 + 4t - 4$$

$$\Rightarrow (D^3-6D^2)y = 0$$

The G.S is ... $y = b_1 + b_2 t + b_3 e^{6t}$

Now, using the initial position $x = x_0$, $y = y_0$, $z = z_0$ and $t = 0$ we get.

$x_0 = a_1 + a_3$ $\qquad$ $y_0 = b_1 + b_3$ $\qquad$ $z_0 = c_1 + c_3$

$\Rightarrow a_1 = x_0 - a_3$ $\qquad$ $\Rightarrow b_1 = y_0 - b_3$ $\qquad$ $\Rightarrow c_1 = z_0 - c_3$

using this we get...

$$x = x_0 - a_3 + a_2 t + a_3 e^{6t} + \frac{1}{4}t^2 \quad \ldots (xix)$$

$$y = y_0 - b_3 + b_2 t + b_3 e^{6t} \quad \ldots (xx)$$

$$z = z_0 - c_3 + c_2 t + c_3 e^{6t} - \frac{1}{4}t^2 \quad \ldots (xxi)$$

putting these values of x, y and z into (iii) (iv) and (v), we get.

$$a_2 + 6a_3 e^{6t} + \frac{1}{2}t = x_0 + y_0 + z_0 - (a_3+b_3+c_3) + (a_2+b_2+c_2)t + (a_3+b_3+c_3)\cancel{t^2} e^{6t} + t$$

$$b_2 + 6b_3 e^{6t} = 2(x_0+y_0+z_0) - 2(a_3+b_3+c_3) + 2(a_2+b_2+c_2)t + 2(a_3+b_3+c_3)e^{6t} + t$$

$$c_2 + 6c_3 e^{6t} - \frac{1}{2}t = 3(x_0+y_0+z_0) - 3(a_3+b_3+c_3) + 3(a_2+b_2+c_2)t + 3(a_3+b_3+c_3)e^{6t} + t$$

Now, equating the co-efficients of t, e^{6t} and constant terms we get...

$$x_0 + y_0 + z_0 - (a_3+b_3+c_3) = a_2 \quad \ldots (a)$$

$$a_3 + b_3 + c_3 = 6a_3 \quad \ldots (b)$$

$$a_2 + b_2 + c_2 + 1 = 1/2 \quad \ldots (c)$$

$$2(x_0+y_0+z_0) - 2(a_3+b_3+c_3) = b_2 \quad \text{--- (d)}$$

$$2(a_3+b_3+c_3) = 6b_3 \quad \text{--- (e)}$$

$$2(a_2+b_2+c_2) + 1 = 0 \quad \text{--- (f)}$$

$$3(x_0+y_0+z_0) - 3(a_3+b_3+c_3) = c_2 \quad \text{--- (g)}$$

$$3(a_3+b_3+c_3) = 6c_3 \quad \text{--- (h)}$$

$$3(a_2+b_2+c_2)+1 = -\frac{1}{2} \quad \text{--- (i)}$$

adding (a) (d), (g) we get...

$$6\left[(x_0+y_0+z_0) - (a_3+b_3+c_3)\right] = a_2+b_2+c_2$$

$$\Rightarrow (x_0+y_0+z_0) - (a_3+b_3+c_3) = \frac{1}{6}\left(-\frac{1}{2}\right) \quad [\text{ from (c) or (f) or (i)}]$$

$$\Rightarrow a_3+b_3+c_3 = x_0+y_0+z_0+\frac{1}{12} \quad \text{--- (J)}$$

using this (b) (e) and (h) gives.

$$a_3 = \frac{1}{6}\left(x_0+y_0+z_0+\frac{1}{12}\right)$$

$$b_3 = \frac{1}{3}\left(x_0+y_0+z_0+\frac{1}{12}\right)$$

$$c_3 = \frac{1}{2}\left(x_0+y_0+z_0+\frac{1}{12}\right)$$

Again using (J), (a), (d) and (g) give...

$$a_2 = -\frac{1}{12}, \quad b_2 = -\frac{1}{6}, \quad c_2 = -\frac{1}{4}$$

putting the values of $a_2, b_2, c_2, a_3, b_3, c_3$ in (xix), (xx) and (xxi) we get-

$$x = \frac{5}{6}x_0 - \frac{1}{6}y_0 - \frac{1}{6}z_0 + \frac{1}{6}\left(x_0+y_0+z_0+\frac{1}{12}\right)e^{6t} - \frac{1}{12}t + \frac{1}{4}t^2 - \frac{1}{72}$$

$$y = -\frac{1}{3}x_0 + \frac{2}{3}y_0 - \frac{1}{3}z_0 + \frac{1}{3}\left(x_0+y_0+z_0+\frac{1}{12}\right)e^{6t} - \frac{1}{6}t - \frac{1}{36}$$

$$z = -\frac{1}{2}x_0 - \frac{1}{2}y_0 + \frac{1}{2}z_0 + \frac{1}{2}\left(x_0+y_0+z_0+\frac{1}{12}\right)e^{6t} - \frac{1}{4}t - \frac{1}{4}t^2 - \frac{1}{24}$$

which give the desire displacement x, y and z in lagrangian system considering the initial positions $x = x_0$, $y = y_0$, $z = z_0$ and $t = 0$